

# Regular Ideal Languages and Their Boolean Combinations

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**Abstract.** We consider ideals and Boolean combinations of ideals. For the regular languages within these classes we give expressively complete automaton models. In addition, we consider general properties of regular ideals and their Boolean combinations. These properties include effective algebraic characterizations and lattice identities.

In the main part of this paper we consider the following deterministic one-way automaton models: unions of flip automata, weak automata, and Staiger-Wagner automata. We show that each of these models is expressively complete for regular Boolean combination of right ideals. Right ideals over finite words resemble the open sets in the Cantor topology over infinite words. An omega-regular language is a Boolean combination of open sets if and only if it is recognizable by a deterministic Staiger-Wagner automaton; and our result can be seen as a finitary version of this classical theorem. In addition, we also consider the canonical automaton models for right ideals, prefix-closed languages, and factorial languages.

In the last section, we consider a two-way automaton model which is known to be expressively complete for two-variable first-order logic. We show that the above concepts can be adapted to these two-way automata such that the resulting languages are the right ideals (resp. prefix-closed languages, resp. Boolean combinations of right ideals) definable in two-variable first-order logic.

## 1 Introduction

The Cantor topology over infinite words is an important concept for classifying languages over infinite words. For example, an  $\omega$ -regular language is deterministic if and only if it is a countable intersection of open sets, cf. [18, Remark 5.1]. There are many other properties of  $\omega$ -languages which can be described using the Cantor topology, see e.g. [12, 15]. Ideals are the finitary version of open sets in the Cantor topology. A subset  $P$  of a monoid  $M$  is a right (resp. left, two-sided) *ideal* if  $PM \subseteq P$  (resp.  $MP \subseteq P$ ,  $MPM \subseteq P$ ). In particular, a language  $L \subseteq A^*$  is a right ideal if  $LA^* \subseteq L$ . A *filter* is the complement of

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\*The last two authors were supported by the German Research Foundation (DFG) under grant DI 435/5-1.

an ideal. Thus over finite words, a language  $L \subseteq A^*$  is a right filter if and only if it is *prefix-closed*, *i.e.*, if  $uv \in L$  implies  $u \in L$ . Prefix-closed languages correspond to closed sets in the Cantor topology. A language  $L \subseteq A^*$  is a two-sided filter if and only if it is *factorial* (also known as *factor-closed* or *infix-closed*), *i.e.*, if  $uvw \in L$  implies  $v \in L$ . Our first series of results gives effective algebraic characterizations of right (resp. left, two-sided) ideal languages and of Boolean combinations of such languages. In addition, we give lattice identities for each of the resulting language classes. As a byproduct, we show that a language is both regular and a Boolean combination of right (resp. left, two-sided) ideals if and only if it is a Boolean combination of regular right (resp. left, two-sided) ideals, *i.e.*, if  $\mathcal{I}$  is the class of right (resp. left, two-sided) ideals and  $\text{REG}$  is the class of regular languages, then  $\text{REG} \cap \mathbb{B}\mathcal{I} = \mathbb{B}(\text{REG} \cap \mathcal{I})$ . Here,  $\mathbb{B}$  denotes the Boolean closure.

The second contribution of this paper consists of expressively complete (one-way) automaton models for right ideals, for prefix-closed languages, for factorial languages, and for Boolean combinations of right ideals. The results concerning ideals and closed languages are straightforward and stated here only to draw a more complete picture. Our main original contribution are automaton models for regular Boolean combinations of right ideals. We always assume that every state in an automaton is reachable from some initial state, *i.e.*, all automata in this paper are accessible.

- A *flip automaton* is an automaton with no transitions from final states to non-final states, *i.e.*, it “flips” at most once from a non-final to a final state. Consequently, every minimal complete flip automaton has at most one final state which has a self-loop for each letter of the alphabet. Paz and Peleg have shown that if a language  $L$  is recognized by a complete deterministic automaton  $\mathcal{A}$ , then  $L$  is a right ideal if and only if  $\mathcal{A}$  is a flip automaton [11]. A language is a regular Boolean combination of right ideals if and only if it is recognized by a union of flip automata (which do not have to be complete).
- An automaton is *fully accepting* if all states are final. A word  $u$  is rejected in a fully accepting automaton  $\mathcal{A}$  if and only if there is no  $u$ -labeled path in  $\mathcal{A}$  which starts in an initial state. Nondeterministic fully accepting automata are expressively complete for prefix-closed languages. Moreover, if a language  $L$  is recognized by a deterministic trim automaton  $\mathcal{A}$ , then  $L$  is prefix-closed if and only if  $\mathcal{A}$  is fully accepting.
- A *path automaton* is an automaton  $\mathcal{A}$  such that all states are both initial and final, *i.e.*, a word  $u$  is accepted by  $\mathcal{A}$  if there exists a  $u$ -labeled path in  $\mathcal{A}$ . Both deterministic and nondeterministic path automata recognize exactly the class of regular factorial languages. This characterization can be implicitly found in the work of Avgustinovich and Frid [1].
- An automaton is *weak* if in each strongly connected component either all states are final or all states are non-final. Any run of a weak automaton flips only a bounded number of times between final and non-final states. Nondeterministic weak automata can recognize all regular languages. On the other hand, if a language  $L$  is recognized by a deterministic automaton  $\mathcal{A}$ , then  $L$  is a Boolean combination of right ideals if and only if  $\mathcal{A}$  is weak. Weak automata have been introduced by Muller, Saoudi, and Schupp [10].
- *Deterministic Staiger-Wagner automata* over infinite words have been used for characterizing  $\omega$ -languages  $L \subseteq A^\omega$  such that both  $L$  and  $A^\omega \setminus L$  are deterministic [16]. Acceptance of a run in a Staiger-Wagner automaton only depends on the set of states visited by the run (but not on their order or their number of occurrences). We show that,

over finite words, deterministic Staiger-Wagner automata are expressively complete for Boolean combinations of right ideals. In particular, deterministic Staiger-Wagner automata and deterministic weak automata accept the same class of languages.

We note that flip automata, fully accepting automata, and weak automata yield effective characterizations of the respective language classes. For example, in order to check whether a deterministic automaton  $\mathcal{A}$  recognizes a Boolean combination of right ideals, it suffices to test if  $\mathcal{A}$  is weak. Moreover, the above automaton models can easily be applied to subclasses of automata such as counter-free automata [9]. This immediately yields results of the following kind: A regular language  $L$  is both star-free and a Boolean combination of right ideals if and only if its minimal automaton is weak and counter-free.

For some classes of languages it is more adequate to use two-way automata. The relation between two-way automata and ideals (resp. closed languages, Boolean combinations of ideals) is more complex than for one-way automata. In the last section, we consider deterministic partially ordered two-way automata (po2dfa). Partially ordered automata are also known as *very weak*, *1-weak*, or *linear* automata. We give restrictions of po2dfa's which define the right ideals (resp. prefix-closed languages, Boolean combinations of right ideals) inside the po2dfa-recognizable languages. The class of languages recognized by po2dfa has a huge number of equivalent characterizations; these include the variety **DA** of finite monoids, two-variable first-order logic, unary temporal logic, unambiguous polynomials, and rankers; see *e.g.* [17, 4]. Some of these characterizations admit natural restrictions which are expressively complete for their ideal (resp. prefix-closed, Boolean combination of ideals) counterparts. We introduce one-pass flip po2dfa (resp. one-pass fully accepting po2dfa, one-pass po2dfa) as expressively complete automaton models for right ideals (resp. prefix-closed languages, Boolean combinations of right ideals) inside the class of po2dfa-recognizable languages. For definitions of these automaton models, we refer the reader to Section 5. The main challenge for each of the above automaton models is showing closure under union and intersection since standard techniques, such as sequentially executing one automaton after the other, cannot be applied. As a complementary result we see that weak one-pass two-way dfa's have the same expressive power as their one-way counterparts, *i.e.*, recognize regular Boolean combinations of right ideals.

## 2 Preliminaries

Throughout this paper,  $A$  is a finite alphabet. The set of finite words over the alphabet  $A$  is denoted by  $A^*$ ; it is the free monoid over  $A$ . The neutral element is the empty word  $\varepsilon$ . The set of nonempty words is  $A^+ = A^* \setminus \{\varepsilon\}$ . If a language  $L \subseteq A^*$  satisfies  $LA^* \subseteq L$  (resp.  $A^*L \subseteq L$ ,  $A^*LA^* \subseteq L$ ), then  $L$  is a *right ideal* (resp. *left ideal*, *two-sided ideal*). If  $L = A^* \setminus K$  for some right (resp. left, two-sided) ideal  $K$ , then  $L$  is *prefix-closed* (resp. *suffix-closed*, *factorial*). Factorial languages are also known as *factor-closed* or *infix-closed*. Boolean combinations consist of complementation, *finite* unions, and *finite* intersections.

*Green's relations* on a monoid  $M$  are defined as follows. For  $x, y \in M$  let  $x \leq_{\mathcal{R}} y$  (resp.  $x \leq_{\mathcal{L}} y$ ,  $x \leq_{\mathcal{J}} y$ ) if there exist  $s, t \in M$  such that  $x = ys$  (resp.  $x = ty$ ,  $x = tys$ ). We set  $x \mathcal{R} y$  if both  $x \leq_{\mathcal{R}} y$  and  $y \leq_{\mathcal{R}} x$ . The relations  $\mathcal{L}$  and  $\mathcal{J}$  are defined similarly involving  $\leq_{\mathcal{L}}$  and  $\leq_{\mathcal{J}}$ , respectively. An element  $x \in M$  is *idempotent* if  $x = x^2$ . In every finite monoid  $M$  there exists a number  $\omega \geq 1$  such that  $x^\omega$  is idempotent for all  $x \in M$ . A homomorphism  $h : A^* \rightarrow M$  recognizes a language  $L \subseteq A^*$  if  $L = h^{-1}(P)$  for some  $P \subseteq M$ , *i.e.*,  $u \in L$  if and only if  $h(u) \in P$ . A monoid  $M$  recognizes  $L$  if there exists a homomorphism  $h : A^* \rightarrow M$

recognizing  $L$ . For every regular language  $L$  there exists a unique minimal finite monoid  $\text{Synt}(L)$  which recognizes  $L$  (and which is effectively computable as the transition monoid of the minimal automaton). It is the *syntactic monoid* of  $L$ , and it is naturally equipped with a recognizing homomorphism  $h_L : A^* \rightarrow \text{Synt}(L)$ , called the *syntactic homomorphism*. A language is regular if and only if its syntactic monoid is finite, see *e.g.* [12].

Lattice identities are a tool for describing classes of languages (these language classes form so-called lattices). Lattice identities can be defined in the general setting of free profinite monoids [6]. In this paper, we only introduce the  $\omega$ -notation. We inductively define  $\omega$ -terms over a set of variables  $\Sigma$ : Every  $x \in \Sigma$  is an  $\omega$ -term; and if  $x$  and  $y$  are  $\omega$ -terms, then so are  $xy$  and  $(x)^\omega$ . For a number  $n \in \mathbb{N}$  and an  $\omega$ -term  $x$ , we define  $x(n)$  inductively by  $(xy)(n) = x(n)y(n)$ ,  $(x^\omega)(n) = x(n)^{n!}$ , and  $x(n) = x$  for  $x \in \Sigma$ , *i.e.*,  $x(n)$  is the word obtained by replacing all exponents  $\omega$  in  $x$  by  $n!$ . Intuitively,  $x^\omega$  is the idempotent element generated by  $x$  with respect to *all* regular languages. A regular language  $L$  satisfies the lattice identity  $x \rightarrow y$  for  $\omega$ -terms  $x$  and  $y$  if there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  and for all homomorphisms  $h : \Sigma^* \rightarrow A^*$  the implication  $h(x(n)) \in L \Rightarrow h(y(n)) \in L$  holds. It satisfies  $x \leftrightarrow y$  if  $x \rightarrow y$  and  $y \rightarrow x$ .

### 3 Ideals and Their Boolean Combinations

Many interesting properties over finite words can be stated as follows: There exists a prefix (resp. suffix, factor) which has some desirable property  $L \subseteq A^*$  and we do not care about subsequent actions. This immediately leads to the right ideal  $LA^*$  (resp. left ideal  $A^*L$ , two-sided ideal  $A^*LA^*$ ). Such languages and their Boolean combinations arise naturally, see *e.g.* [2, 7]. We give effective algebraic characterizations and lattice identities for the regular ideal languages (Proposition 1) and the regular Boolean combinations of ideals (Theorem 2). In the case of ideals, the proof is straightforward and relies on the following simple fact. If  $h : M \rightarrow N$  is a surjective homomorphism between monoids and  $I \subseteq M$  as well as  $J \subseteq N$  are right ideals (resp. left ideals, two-sided ideals), then  $h(I)$  and  $h^{-1}(J)$  are also right ideals (resp. left ideals, two-sided ideals), *i.e.*, ideals are closed under homomorphic and inverse homomorphic images.

**Proposition 1** *Let  $L \subseteq A^*$  be a regular language recognized by a surjective homomorphism  $h : A^* \rightarrow M$  onto a monoid  $M$ . The following are equivalent:*

1.  $L$  is a right ideal (resp. left ideal, two-sided ideal).
2.  $h(L)$  is a right ideal (resp. left ideal, two-sided ideal).
3.  $L$  satisfies the lattice identity  $y \rightarrow yz$  (resp.  $y \rightarrow xy$ ,  $y \rightarrow xyz$ ).

*Proof.* Right ideals (resp. left ideals, two-sided ideals) are closed under surjective homomorphisms and under inverse homomorphisms. Thus (1) and (2) are equivalent. We have  $LA^* \subseteq L$  if and only if for all  $y, z \in A^*$  we have  $y \in L \Rightarrow yz \in L$  if and only if  $L$  satisfies the lattice identity  $y \rightarrow yz$ . This establishes the equivalence of (1) and (3) for right ideals; the argument for left ideals and two-sided ideals is analogous.  $\square$

In particular, property (2) of Proposition 1 yields decidability of whether a given regular language is a (right, left, or two-sided) ideal of  $A^*$  because the syntactic homomorphism  $h_L : A^* \rightarrow \text{Synt}(L)$  and the set  $h_L(L)$  are effectively computable. Moreover, regular (right, left, and two-sided) ideals are closed under union, intersection, and inverse homomorphisms. They do not form so-called *positive varieties* because they are not closed under residuals

(even though right ideals are closed under left residuals, and left ideals are closed under right residuals), cf. [12]. An easy example is  $L = abA^*$  over the alphabet  $A = \{a, b\}$ ; we have  $a \in Lb^{-1} = L \cup \{a\}$  and  $aa \notin Lb^{-1}$ , showing that  $Lb^{-1}$  is not a right ideal.

In the next theorem, we consider Boolean combinations of ideals. Note that if  $h : M \rightarrow N$  is a surjective homomorphism and  $I, J$  are ideals of  $M$ , then in general, we have  $h(I \setminus J) \neq h(I) \setminus h(J)$ . Another obstacle for Boolean combinations of ideals is the following: If  $L$  is regular and a Boolean combination of ideals  $K_i$ , then the  $K_i$  need not be regular. As a byproduct of our characterization in Theorem 2, we see that in the above situation, one can find regular ideals  $K'_i$  such that  $L$  is a Boolean combination of the languages  $K'_i$ .

**Theorem 2** *Let  $L \subseteq A^*$  be a language recognized by a surjective homomorphism  $h : A^* \rightarrow M$  onto a finite monoid  $M$ . Then the following are equivalent:*

1.  $L$  is a Boolean combination of right (resp. left, two-sided) ideals.
2.  $h(L)$  is a union of  $\mathcal{R}$ -classes (resp.  $\mathcal{L}$ -classes,  $\mathcal{J}$ -classes).
3.  $L$  satisfies the lattice identity  $z(xy)^\omega x \leftrightarrow z(xy)^\omega$  (resp. the identity  $s(ts)^\omega z \leftrightarrow (ts)^\omega z$ , the identity  $s(ts)^\omega z(xy)^\omega x \leftrightarrow (ts)^\omega z(xy)^\omega$ ).

*Proof.* We show  $(1) \Leftrightarrow (2)$  and  $(2) \Leftrightarrow (3)$  for right ideals. Left ideals and two-sided ideals are similar. For words  $u, v \in A^*$  we write  $u \equiv v$  if  $h(u) = h(v)$ .

$(1) \Rightarrow (2)$ : Let  $L$  Boolean combination of right ideals. Then  $L = \bigcup_{i=1}^n P_i \setminus Q_i$  for right ideals  $P_i$  and  $Q_i$ . To see this, we first use De Morgan's law in order to move negations inwards so that neither any intersection nor any union is negated. Then we perform an induction on the resulting positive Boolean expression. For right ideals and negations thereof the claim is trivially true. For union the induction step is also clear. Let now  $L = L_1 \cap L_2$  and let  $L_1 = \bigcup_i P_i \setminus Q_i$  and  $L_2 = \bigcup_j P'_j \setminus Q'_j$ . Then  $L = \bigcup_{i,j} (P_i \cap P'_j) \setminus (Q_i \cup Q'_j)$  and the claim follows since right ideals are closed under union and intersection. Consider  $u, v$  such that  $h(u) \mathcal{R} h(v)$  and let  $x, y \in A^*$  such that  $v \equiv ux$  and  $u \equiv vy$ . Suppose  $h(u) \in h(L)$ . Let  $u_j = u(xy)^j$  and  $v_j = v_j x$ . Now,  $u_j \equiv u$ ,  $v_j \equiv v$ , and  $u_j$  is a prefix of  $v_j$  which in turn is a prefix of  $u_{j+1}$ . Every  $u_j$  is in  $L$  and hence for every  $j \in \mathbb{N}$  there exists  $i \in \{1, \dots, n\}$  such that  $u_j \in P_i \setminus Q_i$ . By the pigeonhole principle there exist  $j < k$  with  $u_j, u_k \in P_i \setminus Q_i$  for some  $i \in \{1, \dots, n\}$ . Then  $v_j \in P_i A^* \subseteq P_i$ . If  $v_j \in Q_i$ , then  $u_k \in Q_i A^* \subseteq Q_i$  and  $u_k \notin P_i \setminus Q_i$ , a contradiction. Thus  $v_j \notin Q_i$  and  $v_j \in P_i \setminus Q_i \subseteq L$ . Hence,  $h(v) = h(v_j) \in h(L)$ . This shows that  $h(L)$  is a union of  $\mathcal{R}$ -classes.

$(2) \Rightarrow (1)$ : Let  $R$  be an  $\mathcal{R}$ -class of  $M$ . Consider the two right ideals  $R' = \{x \mid x \leq_{\mathcal{R}} R\}$  and  $R'' = \{x \mid x <_{\mathcal{R}} R\}$ . Then  $h^{-1}(R) = h^{-1}(R') \setminus h^{-1}(R'')$  is a Boolean combination of right ideals (since right ideals are closed under inverse homomorphisms). With  $h(L)$  being a finite union of  $\mathcal{R}$ -classes, the claim follows.

$(2) \Rightarrow (3)$ : Suppose  $h(L)$  is a union of  $\mathcal{R}$ -classes. For every sufficiently large  $n \geq 1$  we have  $h(z(xy)^n) \mathcal{R} h(z(xy)^n x)$  for all  $x, y, z \in A^*$ . Thus  $z(xy)^n \in L \Leftrightarrow z(xy)^n x \in L$ , showing the lattice identity.

$(3) \Rightarrow (2)$ : Suppose  $h(w) \mathcal{R} h(z) \in h(L)$ . Then there exist  $x, y \in A^*$  such that  $z \equiv wy$  and  $w \equiv zx$ . We have  $z \equiv z(xy)^n$  for all  $n \in \mathbb{N}$ . Hence,  $z(xy)^n \in L$ . Choosing  $n$  sufficiently large, the lattice identity yields  $w \equiv z(xy)^n x \in L$  and  $h(w) \in h(L)$ , showing that  $h(L)$  is a union of  $\mathcal{R}$ -classes.  $\square$

Since Theorem 2 (2) can be verified effectively for the syntactic homomorphism, it is decidable whether a given regular language is a Boolean combination of right ideals (resp. left ideals, two-sided ideals).

Every  $\mathcal{R}$ -class is the set difference between two right ideals. Thus if  $L$  is a Boolean combination of (arbitrary) right ideals and if  $L$  is recognized by  $h : A^* \rightarrow M$ , then by Theorem 2, the language  $L$  can also be written as a Boolean combination of right ideals  $K_i$  such that each  $K_i$  is recognized by  $h$ . The situation for Boolean combinations of left ideals (resp. two-sided ideals) is similar.

For finite monoids,  $\mathcal{J}$  is the smallest equivalence relation such that  $\mathcal{R} \subseteq \mathcal{J}$  and  $\mathcal{L} \subseteq \mathcal{J}$ , see e.g. [12, Proposition A.2.5 (2)]. Hence, it follows from Theorem 2 that a regular language  $L$  is a Boolean combination of two-sided ideals if and only if  $L$  is both a Boolean combination of right ideals and a Boolean combination of left ideals.

In Boolean combinations of right ideals, intuitively speaking, what happens is that the end of words is “concealed.” Appending a new symbol as an end-marker to a language yields a Boolean combination of right ideals. Specifically, if  $L$  is language over  $A \setminus \{a\}$ , then  $La$  is a Boolean combination of right ideals of  $A^*$  because  $La = LaA^* \setminus LaA^+$ . In Section 5, we will avoid this “revealing” of the end of the word by the right end marker by considering one-pass automata.

## 4 One-way Automaton Models

As usual, an *automaton*  $\mathcal{A} = (Q, A, \delta, Q_0, F)$  is given by a finite set of states  $Q$ , an input alphabet  $A$ , a transition relation  $\delta \subseteq Q \times A \times Q$ , a set of initial states  $Q_0 \subseteq Q$ , and a set of final states  $F \subseteq Q$ . For transitions  $(p, a, q) \in \delta$  we write  $p \xrightarrow{a} q$  and we inductively extend the transition relation to words:  $q \xrightarrow{\varepsilon} q$  for all  $q \in Q$ ; and  $p \xrightarrow{au} q$  if there exists some  $r \in Q$  such that  $p \xrightarrow{a} r \xrightarrow{u} q$ . A *run* on a word  $a_1 \cdots a_n$  with  $a_i \in A$  is a sequence of states  $q_0 q_1 \cdots q_n$  such that  $q_0 \in Q_0$  and  $q_{i-1} \xrightarrow{a_i} q_i$  for all  $i$ . We always assume that all states are accessible, i.e., for every  $q \in Q$  there exist  $q_0 \in Q_0$  and  $u \in A^*$  such that  $q_0 \xrightarrow{u} q$ . A word  $u \in A^*$  is *accepted* by  $\mathcal{A}$  if there exist  $p \in Q_0$  and  $q \in F$  such that  $p \xrightarrow{u} q$ . The language *recognized* by  $\mathcal{A}$  is  $L(\mathcal{A}) = \{u \in A^* \mid u \text{ is accepted by } \mathcal{A}\}$ . The automaton  $\mathcal{A}$  is *complete* if for every  $p \in Q$  and for every  $a \in A$  there exists at least one state  $q \in Q$  such that  $p \xrightarrow{a} q$ ; it is *trim* if for every  $q \in Q$  there exists  $u \in A^*$  and  $p \in F$  such that  $q \xrightarrow{u} p$ ; and it is *deterministic* if  $|Q_0| = 1$  and for all  $p \in Q$  and all  $a \in A$  there is at most one state  $q \in Q$  with  $p \xrightarrow{a} q$ .

In the remainder of the section, we give automaton models for regular right ideals, prefix-closed languages, factorial languages, and Boolean combinations of right ideals. The results concerning ideals and closed languages are straightforward and presented here only for the sake of completeness. Our main original contribution is Theorem 7, where we give three automaton descriptions of Boolean combinations of ideals: deterministic weak automata, deterministic Staiger-Wagner automata, and unions of deterministic flip automata.

A *flip automaton* is an automaton such that  $p \in F$  and  $p \xrightarrow{a} q$  implies  $q \in F$ . The idea is that, in every run, flip automata can “flip” at most once from non-accepting to accepting. Note that the language of a complete flip automata remains unchanged if we add a self-loop  $q \xrightarrow{a} q$  for every state  $q \in F$  and every letter  $a \in A$ .

**Proposition 3** *Let  $L \subseteq A^*$  be recognized by a complete deterministic automaton  $\mathcal{A}$ . Then the following are equivalent:*

1.  $L$  is a right ideal.
2.  $\mathcal{A}$  is a flip automaton.
3.  $L$  is recognized by some complete (nondeterministic) flip automaton.

*Proof.* (1)  $\Rightarrow$  (2): Let  $\mathcal{A} = (Q, A, \delta, q_0, F)$  and suppose  $p \xrightarrow{a} q$  for  $p \in F$  and  $a \in A$ . Since every state is reachable, there exists a word  $u \in A^*$  such that  $q_0 \xrightarrow{u} p$ . In particular,  $u \in L$ . Since  $L$  is a right ideal, we have  $ua \in L$ . Now,  $q_0 \xrightarrow{ua} q$  yields  $q \in F$ . The implication (2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1): Let  $\mathcal{A}' = (Q, A, \delta, Q_0, F)$  be a complete flip automaton recognizing  $L$ . Suppose  $u \in L$ , and let  $a \in A^*$  be arbitrary. Then  $q_0 \xrightarrow{u} p$  for  $q_0 \in Q_0$  and  $p \in F$ . In addition, since  $\mathcal{A}'$  is complete, we have  $p \xrightarrow{a} q$ . Moreover,  $q \in F$  because  $\mathcal{A}'$  is a flip automaton. This shows  $LA \subseteq L$  and thus  $LA^* \subseteq L$ .  $\square$

The equivalence of (1) and (2) in Proposition 3 is due to Paz and Peleg [11]. Of course, not every complete nondeterministic automaton which recognizes a right ideal has to be a flip automaton. Note that arbitrary (*i.e.*, non-complete and nondeterministic) flip automata can recognize all regular languages.

A *fully accepting automaton* is an automaton in which all states are final, *i.e.*,  $F = Q$ . The only possibility to reject a word is a missing outgoing transition at some point of the computation. Complementing Proposition 3 leads to the following characterization of fully accepting automata.

**Corollary 4** *Let  $L \subseteq A^*$  be recognized by a deterministic trim automaton  $\mathcal{A}$ . Then the following are equivalent:*

1.  $L$  is prefix-closed.
2.  $\mathcal{A}$  is fully accepting.
3.  $L$  is recognized by some (nondeterministic) fully accepting automaton.

*Proof.* (1)  $\Rightarrow$  (2): Let  $\mathcal{A} = (Q, A, \delta, q_0, F)$  and assume  $p \in Q \setminus F$ . Since  $\mathcal{A}$  is trim, there exist  $q \in F$  and  $u, v \in A^*$  such that  $q_0 \xrightarrow{u} p \xrightarrow{v} q$ . Now,  $uv \in L$  implies  $u \in L$  and  $p \in F$ , a contradiction.

The implication (2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1): Let  $\mathcal{A}' = (Q, A, \delta, Q_0, F)$  be a nondeterministic fully accepting automaton recognizing  $L$ . Suppose  $u \notin L$ , *i.e.*, for every  $q_0 \in Q_0$  and  $q \in Q$  we have  $(q_0, u, q) \notin \delta$ . Thus for every  $v \in A^*$  and every  $q_0 \in Q_0$  and  $q \in Q$  we have  $(q_0, uv, q) \notin \delta$ , *i.e.*, we have  $uv \notin L$ . This shows that  $A^* \setminus L$  is a right ideal.  $\square$

A *path automaton* is an automaton such that every state is both initial and final, *i.e.*,  $Q_0 = F = Q$ . In particular, a path automaton accepts a word  $u \in A^*$  if and only if there exists a path  $p \xrightarrow{u} q$  for some  $p, q \in Q$ .

**Corollary 5** *Let  $L \subseteq A^*$  be a regular language. Then  $L$  is factorial if and only if  $L$  is recognized by a path automaton.*

*Proof.* “ $\Rightarrow$ ”: By Corollary 4 (and since  $L$  is prefix-closed), the language  $L$  is recognized by a deterministic fully accepting automaton  $\mathcal{A} = (Q, A, \delta, q_0, Q)$ . We show  $L(Q, A, \delta, Q, Q) \subseteq L$ . Suppose  $p \xrightarrow{v} q$  for some  $v \in A^*$ . Then there exists  $u \in A^*$  such that  $q_0 \xrightarrow{u} p$ , *i.e.*,  $uv \in L$ . Since  $L$  is suffix-closed, we have  $v \in L$ .

“ $\Leftarrow$ ”: Let  $\mathcal{A} = (Q, A, \delta, Q, Q)$  be a path automaton with  $L(\mathcal{A}) = L$ . Then  $\mathcal{A}$  as well as the automaton obtained by reversing all edges (and interchanging initial and final states – which in this case has no effect) are fully accepting. Thus, by Corollary 4, both the language  $L$  and its reversal  $L^r = \{a_1 \cdots a_n \in A^* \mid a_i \in A, a_n \cdots a_1 \in L\}$  are prefix-closed. It follows that  $L$  is factorial.  $\square$

For deterministic transition relations, the statement of Corollary 5 can be found implicitly in the work of Avgustinovich and Frid [1].

An automaton is *weak* if for every strongly connected component  $C \subseteq Q$ , we either have  $C \subseteq F$  or  $C \cap F = \emptyset$ . The concept of weak automata has been introduced by Muller, Saoudi, and Schupp [10] for alternating tree automata. A *Staiger-Wagner automaton* is given by  $\mathcal{B} = (Q, A, \delta, q_0, \mathcal{T})$  where  $\mathcal{T} \subseteq 2^Q$ . Acceptance of a run by a Staiger-Wagner automaton only depends on the set of states visited by the run. A run  $q_0q_1 \cdots q_n$  is *accepting* if  $\{q_0, q_1, \dots, q_n\} \in \mathcal{T}$ ; and a word is accepted if it has an accepting run.

**Lemma 6** *Let  $\mathcal{A} = (Q, A, \delta, Q_0, F)$  be a weak automaton. Then there exists  $\mathcal{T}$  such that the Staiger-Wagner automaton  $\mathcal{B} = (Q, A, \delta, Q_0, \mathcal{T})$  recognizes  $L(\mathcal{A})$ .*

*Proof.* Let  $\text{future}(q)$  denote the set of states which are reachable from  $q$  and which are not located in the same strongly connected component as  $q$ . We can construct  $\mathcal{T}$  as follows:

$$\mathcal{T} = \{T \mid \exists q \in F \cap T: T \subseteq Q \setminus \text{future}(q)\}.$$

Each element of  $\mathcal{T}$  guarantees, that a run ends within an accepting strongly connected component of  $\mathcal{A}$ . Since  $\mathcal{A}$  is weak, we conclude  $L(\mathcal{A}) = L(\mathcal{B})$ .  $\square$

Our next result shows that both deterministic weak automata and deterministic Staiger-Wagner automata are expressively complete for Boolean combinations of right ideals. Moreover, if a deterministic automaton  $\mathcal{A}$  recognizes a Boolean combination of right ideals, then, by Lemma 6, the automaton  $\mathcal{A}$  itself can be equipped with a Staiger-Wagner acceptance condition. A third automaton model for Boolean combinations of right ideals is given by unions of (not necessarily complete) deterministic flip automata. This last property follows from Theorem 2 since the inverse homomorphic image of every  $\mathcal{R}$ -class of a finite monoid is recognizable by a flip automaton.

**Theorem 7** *Let  $L \subseteq A^*$  be recognized by a deterministic automaton  $\mathcal{A}$ . Then the following are equivalent:*

1.  $L$  is a Boolean combination of right ideals.
2.  $\mathcal{A}$  is weak.
3.  $L$  is recognized by some deterministic Staiger-Wagner automaton.
4.  $L$  is a finite disjoint union of languages  $L(\mathcal{B}_i)$  such that each  $\mathcal{B}_i$  is a deterministic flip automaton.

*Proof.* (1)  $\Rightarrow$  (2): Let  $\mathcal{A} = (Q, A, \delta, q_0, F)$ . Assume  $p \xrightarrow{x} q$  and  $q \xrightarrow{y} p$  for  $p \in F$ . Choose  $z \in A^*$  such that  $q_0 \xrightarrow{z} p$ . Then for all  $n \in \mathbb{N}$  we have  $z(xy)^n \in L$ . By Theorem 2, the language  $L$  satisfies the lattice identity  $z(xy)^\omega \leftrightarrow z(xy)^\omega x$ . Therefore, for some  $n$ , we have  $z(xy)^n x \in L$ . Now,  $\delta(q_0, z(xy)^n x) = q$  implies  $q \in F$ .

The implication (2)  $\Rightarrow$  (3) follows by Lemma 6.

(3)  $\Rightarrow$  (1): Let  $\mathcal{B} = (Q, A, \delta, q_0, \mathcal{T})$  be a deterministic Staiger-Wagner automaton. We show that  $L(\mathcal{B})$  satisfies the lattice identity  $z(xy)^\omega \leftrightarrow z(xy)^\omega x$ . Let  $x, y, z \in A^*$  and let  $n \geq |Q|$ . Let  $q_0 \xrightarrow{z} q_1$  and  $q_i \xrightarrow{xy} q_{i+1}$  for  $1 \leq i \leq n$ . By choice of  $n$  there exist  $1 \leq k < \ell \leq n+1$  such that  $q_k = q_\ell$ . It follows that, for all  $m \in \mathbb{N}$ , the runs of the words  $z(xy)^{\ell-1}(xy)^m$  and  $z(xy)^{\ell-1}(xy)^m x$  both visit the same states as  $z(xy)^{\ell-1}$ . In particular,  $z(xy)^n \in L$  if and only if  $z(xy)^n x \in L$  (which proves the lattice identity  $z(xy)^\omega \leftrightarrow z(xy)^\omega x$  for  $L$ ).

(2)  $\Rightarrow$  (4): Let  $\mathcal{A} = (Q, A, \delta, q_0, F)$  be weak. For a strongly connected component  $C \subseteq F$  we define  $\mathcal{B}_C = (Q_C, A, \delta_C, q_0, F \cap C)$  as the (not necessarily complete) flip automaton



with states  $Q_C = \{q \in Q \mid C \text{ is reachable from } q\}$ , and its transition function  $\delta_C$  is the restriction of  $\delta$  to states  $Q_C$ . Then  $L(\mathcal{A}) = \bigcup_C L(\mathcal{B}_C)$  where the union ranges over all strongly connected components  $C \subseteq F$ . Note that this union is indeed disjoint because  $\mathcal{A}$  is deterministic.

(4)  $\Rightarrow$  (2): Every flip automaton is weak. Moreover, languages recognized by weak automata are closed under union by the equivalence of (1) and (2).  $\square$

Both nondeterministic weak automata and nondeterministic Staiger-Wagner automata are expressively complete for the class of all regular languages as the next lemma shows. In particular, the nondeterministic variants of weak automata and Staiger-Wagner automata do *not* characterize Boolean combinations of right ideals.

**Lemma 8** *Let  $L \subseteq A^*$ . The following are equivalent:*

1.  $L$  is regular.
2.  $L$  is recognized by a (nondeterministic) weak automaton.
3.  $L$  is recognized by a (nondeterministic) Staiger-Wagner automaton.

*Proof.* (1)  $\Rightarrow$  (2): Let  $L$  be recognized by the deterministic automaton  $\mathcal{A} = (Q, A, \delta, q_0, F)$  and let  $f \notin Q$  be a new state. Let

$$\delta' = \delta \cup \{(p, a, f) \mid (p, a, q) \in \delta \text{ and } q \in F\}.$$

We set  $Q_0 = \{q_0, f\}$  if  $q_0 \in F$ , and  $Q_0 = \{q_0\}$  otherwise. This way we introduce a single accepting state  $f$  which can be reached nondeterministically if and only if there was a path from the initial state to some final state in  $\mathcal{A}$ . Thus  $(Q, A, \delta', Q_0, \{f\})$  recognizes  $L$ .

(2)  $\Rightarrow$  (3): follows from Lemma 6.

(3)  $\Rightarrow$  (1): Let  $\mathcal{B} = (Q, A, \delta, Q_0, \mathcal{T})$  be a nondeterministic Staiger-Wagner automaton. We can construct  $\mathcal{A} = (2^Q \times Q, A, \delta', Q'_0, F)$  with  $((P, q), a, (P', q')) \in \delta'$  if and only if  $(q, a, q') \in \delta \wedge P' = P \cup \{q'\}$ . The set of initial states is  $Q'_0 = \{(\{q\}, q) \mid q \in Q_0\}$ , and the set of final states is defined by  $(P, q) \in F$  if and only if  $P \in \mathcal{T}$ . This way,  $\mathcal{A}$  simulates  $\mathcal{B}$  along each path and collects the visited states. It accepts, if the set of visited states is in  $\mathcal{T}$ . Therefore  $L(\mathcal{A}) = L(\mathcal{B})$ .  $\square$

**Remark 9** *Proposition 3 (resp. Corollary 4, Theorem 7) yields another decision procedure for the class of regular right ideals (resp. prefix-closed languages, Boolean combinations of right ideals). In the case of Proposition 3, this was first observed by Paz and Peleg [11]. Moreover, the above decidability results can often be combined with other automaton models. For example, a well-known result of McNaughton and Papert says that a language is definable in first-order logic if and only if its minimal automaton is counter-free [9]. Together with Theorem 7, we see that a language  $L$  is a first-order definable Boolean combination of right ideals if and only if the minimal automaton of  $L$  is weak and counter-free.  $\diamond$*

## 5 Two-way Automaton Models and Languages in $\mathcal{DA}$

The results in the previous section can easily be translated into characterizations of regular left ideals (resp. suffix-closed languages, Boolean combinations of left ideals) by considering automata which read the input from right to left. Varying the direction of the head movement naturally leads to two-way automata. The situation for arbitrary two-way automata is more

involved than for one-way automata; the main reason is that two-way automata are usually defined using left and right end markers. On the other hand, if  $L \subseteq (A \setminus \{a\})^*$ , then  $La = LaA^* \setminus LaA^+$ . This shows that by adding an explicit end marker, every language becomes a Boolean combination of right ideals. To overcome this, we introduce the notion of *one-pass* two-way automata; these automata stop processing the input as soon as they read the right end marker. Now, the problem with classes of one-pass two-way automata is that, in general, they may not be closed under union and intersection (standard techniques, such as executing one automaton after the other, cannot be applied). We have no satisfactory solution for arbitrary two-way automata, but we show that the concepts of Section 4 can be adapted to a well-known subclass of two-way automata, namely deterministic partially ordered two-way automata (po2dfa). The class of languages recognized by po2dfa is a natural subclass of the star-free languages which has a huge number of different characterizations, see *e.g.* [17, 4]. The most prominent of these characterizations is definability in two-variable first-order logic. By a description of algebraic means, it is the language variety  $\mathcal{DA}$ , *i.e.*, the class of regular languages satisfying the lattice identity  $p(xy)^\omega q \leftrightarrow p(xy)^\omega x(xy)^\omega q$ . As a byproduct, we show that some of the other characterizations of po2dfa recognizable languages also admit natural counterparts for right ideals and their Boolean combinations.

A *two-way automaton* is a tuple  $\mathcal{A} = (Z, A, \delta, X_0, F)$ . The finite set of states  $Z = X \dot{\cup} Y$  is partitioned into *right-moving* states  $X$  (for neXt) and *left-moving* states  $Y$  (for Yesterday). The states in  $X_0 \subseteq X$  are initial and states in  $F \subseteq Z$  are final. On input  $u \in A^*$ , the tape content is  $\triangleright u \triangleleft$  where  $\triangleright$  and  $\triangleleft$  are new symbols marking the left and right end of the tape, respectively. Initially, the head is at the first letter of  $u$ . The direction in which the input is processed can be controlled by  $\mathcal{A}$ . The idea is that *before* a transition is made, the head movement is performed, and the direction of the movement depends only on the *destination state* of the transition. The left end marker  $\triangleright$  must not be overrun. More formally, the transition relation satisfies  $\delta \subseteq (Z \times A \times Z) \cup (Y \times \{\triangleright\} \times X) \cup (X \times \{\triangleleft\} \times Z)$ . As for one-way automata, we write  $z \xrightarrow{a} z'$  instead of  $(z, a, z') \in \delta$ . More formally, a *configuration* is a pair  $(z, i) \in Z \times \mathbb{N}$  where  $z$  is the current state and  $i$  is the current position on the tape. Suppose position  $i$  is labeled by  $a \in A \cup \{\triangleright, \triangleleft\}$ . Then a transition  $(z, i) \vdash_{\mathcal{A}} (z', j)$  between configurations exists if  $z \xrightarrow{a} z'$  and  $j = i + 1$  (for  $z' \in X$ ) or  $j = i - 1$  (for  $z' \in Y$ ). A *computation* of  $\mathcal{A}$  on input  $u$  is a sequence

$$(z_0, i_0) \vdash_{\mathcal{A}} \cdots \vdash_{\mathcal{A}} (z_t, i_t)$$

of configurations such that  $z_0 \in X_0$ ,  $i_0 = 1$ ,  $i_k \in \{0, \dots, |u| + 1\}$  for  $1 \leq k < t$ , and  $i_t = |u| + 2$ . Note that position 0 is labeled with the left end marker  $\triangleright$  and the position  $|u| + 1$  is labeled with the right end marker  $\triangleleft$ . The computation is *accepting* if  $z_t \in F$  is final and the input  $u$  is accepted if there exists an accepting computation for it. Note that by the signature of the transition relation, the left end marker  $\triangleright$  cannot be trespassed. One-way automata may be seen as special cases with  $Y = \emptyset$ . The language  $L(\mathcal{A})$  *recognized* by  $\mathcal{A}$  is  $L(\mathcal{A}) = \{u \in A^* \mid \mathcal{A} \text{ accepts } u\}$ .

A two-way automaton is *deterministic* if  $|X_0| = 1$  and for all  $z \in Z$  and all  $a \in A \cup \{\triangleright, \triangleleft\}$  there exists at most one  $z' \in Z$  with  $z \xrightarrow{a} z'$ . For technical reasons, we also consider the empty automaton ( $Z = \delta = X_0 = F = \emptyset$ ) as deterministic. It is *complete* if for all  $z \in Z$  and all  $a$  there exists  $z' \in Z$  with  $z \xrightarrow{a} z'$  (more precisely, we require the existence of  $z'$  if either  $z \in Y$  and  $a \in A \cup \{\triangleright\}$  or if  $z \in X$  and  $a \in A \cup \{\triangleleft\}$ ). A two-way automaton is *one-pass* if  $z \xrightarrow{a} z'$  implies  $z = z'$ . The idea is that a two-way automaton has finished “one pass” when it encounters the right end marker  $\triangleleft$  for the first time; hence for a one-pass automaton, the

acceptance of a word is determined by the state when scanning  $\triangleleft$  for the first time. The automaton is *partially ordered* if there exists a partial ordering  $\sqsubseteq$  of the states such that transitions are non-descending, *i.e.*, if  $z \xrightarrow{a} z'$ , then  $z \sqsubseteq z'$ . In other words, once a state is left in a partially ordered automaton, it is never re-entered. We abbreviate “deterministic partially ordered two-way automaton” by *po2dfa*.

Schwentick, Thérien, and Vollmer [14] have shown that po2dfa are expressively complete for  $\mathcal{DA}$ . The main result of this section is a characterization of Boolean combinations of right ideals (resp. right ideals, prefix-closed languages) in  $\mathcal{DA}$  in terms of subclasses of one-pass po2dfa.

A crucial property of one-pass po2dfa is closure under Boolean combinations; and to see this, we shall need the following synchronization lemma. The same lemma was formulated already in [8, Lemma 8] for Büchi automata and infinite words. The *alphabet* of a word  $u$  is denoted by  $\text{alph}(u)$ . The  $i$ th letter of  $u$  is  $u(i)$ .

**Lemma 10 (Synchronization Lemma)** *Consider a po2dfa  $\mathcal{A}$  with states  $Z = X \dot{\cup} Y$ . For every  $v = a_1 \cdots a_m \in \Gamma^+$  there exists a po2dfa  $\mathcal{C}$  with states  $Z_{\mathcal{C}} = Z \times \{v\} \times \{1, \dots, m\}$  such that, for all  $u \in \Gamma^*$  having a factorization  $u = u_1 a_1 \cdots u_m a_m u'$  with  $a_i \notin \text{alph}(u_i)$ , the following simulation property holds: If*

$$(z_0, i_0) \vdash_{\mathcal{A}} (z_1, i_1) \vdash_{\mathcal{A}} \cdots \vdash_{\mathcal{A}} (z_n, i_n)$$

*is a sequence of transitions of  $\mathcal{A}$  for some  $n \geq 1$  with  $i_0 = i_n = |u_1 a_1 \cdots u_m a_m|$  and  $i_t < i_n$  for all  $1 \leq t < n$ , then*

$$((z_1, v, k_1), i_1) \vdash_{\mathcal{C}} \cdots \vdash_{\mathcal{C}} ((z_n, v, k_n), i_n)$$

*is a sequence of transitions of  $\mathcal{C}$  with  $k_1 = k_n = m$  such that there exists no  $1 \leq t < n$  with  $z_t \in X$ ,  $k_t = m$ , and  $u(i_t) = a_m$ .  $\square$*

Intuitively, this means that if a deterministic po2-automaton moves left at some point in its computation, then it may recognize the position on the input *on-the-fly*—provided that this happens at a suitable position, *i.e.*, the  $a_m$  in the factorization stipulated in Lemma 10. In the latter application, determinism will yield such a factorization and for a partially ordered automaton the parameter  $m$  can be bounded over all inputs  $u \in A^*$ . Note that [8, Lemma 8] was formulated with Büchi automata on infinite words. However, the acceptance condition does not influence the statement at all and, since the computations considered in the lemma take place completely on the finite prefix  $u_1 a_1 \cdots u_m a_m$ , the behavior of the automata is independent of the suffix  $u'$  which may even be an infinite word.

**Lemma 11** *The class of languages recognizable by one-pass po2dfa is a Boolean algebra.*

*Proof.* Suppose that  $\mathcal{A} = (X \dot{\cup} Y, A, \delta, x_0, F)$  is a one-pass po2dfa. By adjoining a new non-final right-moving sink state, we may assume that  $\mathcal{A}$  is complete. Then  $\mathcal{A}' = (X \dot{\cup} Y, A, \delta, x_0, X \setminus F)$  recognizes  $A^* \setminus L(\mathcal{A})$ . Therefore, one-pass po2dfa are closed under complementation. It remains to show closure under union.

We describe a product automaton construction for the union of two automaton which executes both automata in parallel. Of course, there is only one head to do this, and the main problem to overcome in this construction is when the automata disagree on the head movement. We shall only give a high-level description of the construction; details can be implemented similarly to the situation for deterministic po2-Büchi automata [8].

By adding a new sink state as needed, we may assume that both automata are complete. The two automata are simulated in parallel as long as they agree on moving to the right. This is called the *synchronous mode*. If at least one of the automata changes to left-moving, then we start a simulation of one of the left-moving automata in the so-called *asynchronous mode* while suspending the other automaton. We refer to the position of the input where this divergence happens as the *synchronization point*. In asynchronous mode, the active automaton can move in either direction. As soon as the synchronization point is reached again and both automata agree on moving to the right, we switch back to synchronous mode and continue simulating both automata in parallel; otherwise we stay in asynchronous mode while simulating one of the automata. To implement this idea, the synchronization point must be recognized while in the asynchronous mode.

For this re-synchronization, we use Lemma 10 and some combinatorial property of computations of po2dfa. Assume that we are about to enter the asynchronous mode. Suppose the input  $u$  is factorized  $u = u_1 a_1 \cdots u_m a_m u'$  such that the  $a_i$ 's correspond to the positions where during synchronous mode at least one of the automata changed its state. Note that  $a_m$  corresponds to the synchronization point because a change from right-moving to left moving implies a change of state. By determinism, we have  $a_i \notin \text{alph}(u_i)$ . Moreover, since both automata are partially ordered,  $m$  is bounded by the sum of the number of states of both automata. The last observation allows to store the word  $v = a_1 \cdots a_m$  in a bounded stack of letters in the state space. Using the automaton from Lemma 10 for  $v$  as the active automaton, we can simulate the active automaton whilst being aware of whenever the synchronization point is reached again. Both automata are complete and thus the synchronization point is eventually reached by the active automaton. After this, we switch back to synchronous mode to simulate both automata in parallel. In synchronous mode the stack of letters is administered, *i.e.*, whenever a state change happens in one of the automata whilst in synchronous mode, the currently scanned letter is pushed to the stack. At the end, we accept if one of the automata accepts.

The procedure given above can be done effectively in such a way that the simulating automaton is a complete, deterministic one-pass po2-automata. The actual construction is along the lines of the proof of [8, Proposition 9] and therefore not given here.  $\square$

A *monomial* is a language  $P = A_1^* a_1 \cdots A_k^* a_k A_{k+1}^*$  where  $A_i \subseteq A$  and  $a_i \in A$ . It is *unambiguous* if every word  $u \in P$  has a unique factorization  $u = u_1 a_1 \cdots u_k a_k u_{k+1}$  with  $u_i \in A_i^*$ . A convenient intermediate step from languages in  $\mathcal{DA}$  to automata are rankers. A *ranker* is a word in  $\{X_a, Y_a \mid a \in A\}^*$ . Intuitively, a ranker  $r$  represents a sequence of instructions  $X_a$  for “next  $a$ -position” and  $Y_a$  for “previous  $a$ -position” which is processed from left to right. That is, for a word  $u = a_1 \cdots a_n$  with  $a_j \in A$  and a position  $i \in \{0, \dots, n+1\}$  we set  $\varepsilon(u, i) = i$  and

$$\begin{aligned} X_a r(u, i) &= r(u, \min \{j > i \mid a_j = a\}), \\ Y_a r(u, i) &= r(u, \max \{j < i \mid a_j = a\}). \end{aligned}$$

If a nonempty ranker  $r$  starts with an  $X_a$ -modality, then we say that  $r$  is an  $X$ -ranker; and we define  $r(u) = r(u, 0)$ , *i.e.*, the evaluation of  $X$ -rankers starts at the beginning of the word  $u$ . Symmetrically, if  $r$  starts with  $Y_a$ , then  $r(u) = r(u, n+1)$ . As usual,  $\min \emptyset$  and  $\max \emptyset$  are undefined. Thus a nonempty ranker  $r$  either defines a unique position  $r(u)$  in a word  $u$  or  $r(u)$  is undefined. For example,  $X_a Y_b X_c(bac) = 3$  whereas  $X_a Y_b X_c(cba)$  is undefined. For a ranker  $r$  we set  $L(r) = \{u \in A^* \mid r(u) \text{ is defined}\}$ .

**Theorem 12** *Let  $L \subseteq A^*$ . The following are equivalent:*

1.  $L \in \mathcal{DA}(A^*)$  is a Boolean combination of right ideals.
2.  $L$  is a finite union of unambiguous monomials  $A_1^*a_1 \cdots A_k^*a_kA_{k+1}^*$  with  $\{a_i, \dots, a_k\} \not\subseteq A_i$  for all  $i \in \{1, \dots, k\}$ .
3.  $L$  is Boolean combination of languages  $L(r)$  for  $\mathbf{X}$ -rankers  $r$ .
4.  $L$  is recognized by a one-pass po2dfa.

*Proof.* Before turning to the actual proof, we give a rough overview of the techniques employed. Right ideals are the finitary version of open sets in the Cantor topology over infinite words. It is therefore not surprising that a large part of Theorem 12 reduces to infinite words: The proof of the implication from (1) to (2) relies on a result of Diekert and Kufleitner [5, Theorem 6.6]. The step from (2) to (3) uses a characterization of  $\mathbf{X}$ -rankers over infinite words [3, Theorem 3]. Showing the implication from (3) to (4) is the most technical part. In particular, one has to show that one-pass po2dfa are closed under union and intersection. Here, the respective result for po2-Büchi automata cannot be applied directly, but showing closure under union and intersection resembles techniques which were developed for deterministic po2-Büchi automata [8]. This is Lemma 11. Finally, the step from (4) back to (1) easily follows by combining the characterization of po2dfa due to Schwentick, Thérien, and Vollmer [14, Theorem 3.1] with Theorem 2. We need to introduce some more notation for the proof.

A monomial  $A_1^*a_1 \cdots A_k^*a_kA_{k+1}^*$  is *restricted* if there exists no  $i \in \{1, \dots, k\}$  such that  $\{a_i, \dots, a_k\} \subseteq A_i$ . Let  $\mathbf{DA}$  be the class of finite monoids satisfying the identity  $(xy)^\omega = (xy)^\omega x(xy)^\omega$ . A language  $L$  is contained in  $\mathcal{DA}$  if and only if it is recognized by a homomorphism  $h : A^* \rightarrow M$  to a finite monoid in  $\mathbf{DA}$ . The set of finite and infinite words over  $A$  is  $A^\infty$ . The  $\omega$ -iteration of a language  $L \subseteq A^*$  of finite words is  $L^\omega = \{u_1u_2 \cdots \in A^\infty \mid u_i \in L\}$ ; in particular we stipulate the convention  $\varepsilon^\omega = \varepsilon$ . Note that it will always be clear from the context whether by “ $\omega$ ” we mean an infinite product or a generated idempotent. Let  $h : A^* \rightarrow M$  be a homomorphism to a finite monoid  $M$ . For  $x \in M$  let  $[x] = h^{-1}(x)$ . A language  $K \subseteq A^\infty$  of finite and infinite words is recognized by  $h$  if

$$K = \bigcup \{[s][e]^\omega \mid [s][e]^\omega \cap L \neq \emptyset \text{ and } s = se, e^2 = e\}.$$

Note that  $[1]^\omega$  also contains finite words. The evaluation of an  $\mathbf{X}$ -ranker  $r$  extends naturally to infinite words and the  $L(r)$  over  $A^\infty$  consists of all finite or infinite words on which  $r$  is defined. For more details we refer to [3].

(1)  $\Rightarrow$  (2): By Theorem 2 the language  $L \subseteq A^*$  is recognized by some  $h : A^* \rightarrow M \in \mathbf{DA}$  such that  $h(L)$  is a union of  $\mathcal{R}$ -classes. Consider the language

$$K = \bigcup \{[s][e]^\omega \mid [s] \subseteq L \text{ and } s = se, e^2 = e\}$$

of finite and infinite words. We have  $K \cap A^* = L$  and  $K$  is recognized by  $h$ . Consider  $s, t, e, f \in M$  such that  $s = se, t = tf, e^2 = e$  and  $f^2 = f$ . Then because  $h(L)$  is a union of  $\mathcal{R}$ -classes,  $s \mathcal{R} t$  implies  $[s][e]^\omega \subseteq K$  if and only if  $[t][f]^\omega \subseteq K$ . The language  $K$  is a finite union of restricted unambiguous monomials  $A_1^*a_1 \cdots A_k^*a_kA_{k+1}^\infty$  over  $A^\infty$ , see [5, Theorem 6.6]. Therefore,  $L = K \cap A^*$  is a finite union of restricted unambiguous monomials  $A_1^*a_1 \cdots A_k^*a_kA_{k+1}^*$  over  $A^*$ .

(2)  $\Rightarrow$  (3): Let  $L$  be a finite union of restricted unambiguous monomials of the form  $A_1^*a_1 \cdots A_k^*a_kA_{k+1}^*$ . Let  $K \subseteq A^\infty$  be obtained by replacing these monomials by the monomial  $A_1^*a_1 \cdots A_k^*a_kA_{k+1}^\infty$ . Then  $K$  is a union of restricted unambiguous monomials over  $A^\infty$ .

Now,  $K$  is definable over  $A^\infty$  in the first-order fragment  $\Delta_2[<]$ , see [5, Theorem 6.6]. Thus  $K$  is a Boolean combination of languages  $L(r)$  over  $A^\infty$  for  $X$ -rankers  $r$ , see [3, Theorem 3]. It follows that  $L = K \cap A^*$  is a Boolean combination of languages  $L(r)$  over  $A^*$  for  $X$ -rankers  $r$ .

(3)  $\Rightarrow$  (4): It is easy to see that every language  $L(r)$  for an  $X$ -ranker  $r$  is recognizable by a one-pass po2dfa. With Lemma 11, we get closure of one-pass po2dfa under Boolean operations.

(4)  $\Rightarrow$  (1): Let  $L$  be recognized by a complete one-pass po2dfa  $\mathcal{A}$ . In particular,  $\mathcal{A}$  is a po2dfa and thus  $L \in \mathcal{DA}$ , cf. [14, Theorem 3.1]. Let  $n$  be a number greater than the number of states of  $\mathcal{A}$  and let  $x, y, z \in A^*$ . We claim that  $z(xy)^n \in L$  if and only if  $z(xy)^n x \in L$ : Consider the run of  $\mathcal{A}$  on either word. Let  $q$  be the state in which  $\mathcal{A}$  leaves the prefix  $z(xy)^n$  for the first time. Note that this must happen eventually since  $\mathcal{A}$  is complete and the left end marker  $\triangleright$  cannot be trespassed. Then  $q$  is right-moving and  $q \xrightarrow{a} q$  is a loop for all letters in  $a \in \text{alph}(xy)$  by choice of  $n$ . Hence,  $\mathcal{A}$  encounters the right end marker  $\triangleleft$  in the state  $q$  on both inputs  $z(xy)^n$  and  $z(xy)^n x$ . Therefore,  $z(xy)^n$  is accepted if and only if  $z(xy)^n x$  is accepted. By Theorem 2, the language  $L$  is a Boolean combination of right ideals.  $\square$

It is decidable whether a given regular language belongs to  $\mathcal{DA}$ . Therefore, using Proposition 1 and Theorem 2, it is decidable whether a regular language is recognized by an arbitrary (resp. flip, fully final) one-pass po2dfa. The temporal logic version of  $X$ -rankers is denoted  $\text{TL}_X[X_a, Y_a]$ , cf. [3]; it is a fragment of deterministic unary temporal logic  $\text{TL}[X_a, Y_a]$  over the modalities  $X_a$  and  $Y_a$ . The logic  $\text{TL}[X_a, Y_a]$  is expressively complete for  $\mathcal{DA}$ , and  $\text{TL}_X[X_a, Y_a]$  defines the right ideals in  $\mathcal{DA}$ .

**Remark 13** *We use the shortcut “nfa” for nondeterministic finite automaton, and “po1” for partially ordered one-way. Using this notation, we have the following inclusions between language classes recognizable by partially ordered automata:*

$$\text{po1dfa} \subsetneq \text{one-pass po2dfa} \subsetneq \text{po2dfa} \subsetneq \text{po2nfa} = \text{po1nfa}.$$

*The following (very similar) languages show that the inclusions are strict. The language  $\{a, c\}^* ab \{a, b, c\}^*$  is recognizable by some one-pass po2dfa but not by a po1dfa. The language  $\{a, b, c\}^* ab \{b, c\}^*$  is recognizable by a po2dfa but not by any one-pass po2dfa. Finally, the language  $\{a, b, c\}^* ab \{a, b, c\}^*$  is recognizable by some po1nfa but not by any po2dfa. The equivalence of po2nfa and po1nfa is due to Schwentick, Thérien, and Vollmer [14]. For each of the above language classes the membership problem is decidable: The class po1dfa corresponds to  $\mathcal{R}$ -trivial monoids [14], one-pass po2dfa correspond to  $\mathcal{R}$ -classes of monoids in  $\mathbf{DA}$  (Theorem 2 and Theorem 12). The algebraic equivalent of po2dfa is the variety of finite monoids  $\mathbf{DA}$  [14], and po2nfa are expressively complete for the level 3/2 of the Straubing-Thérien hierarchy [14] which is decidable by a result of Pin and Weil [13].  $\diamond$*

In analogy to Theorem 12, there is also an expressively complete two-way automaton model for Boolean combinations of right ideals. A two-way automaton is *weak* if for every strongly connected component either all states are final or all states are non-final. Note that every partially ordered automaton is weak. The following result is our only general result for arbitrary (not partially ordered) deterministic two-way automata.

**Proposition 14** *A regular language is a Boolean combination of right ideals if and only if it is recognized by a deterministic weak one-pass two-way automaton.*

*Proof.* If  $L$  is a Boolean combination of right ideals, then  $L$  is recognized by a deterministic weak (one-way) automaton by Theorem 7. Note that every one-way automaton can also be seen as a two-way automaton without left-moving states.

For the converse, consider a complete deterministic weak one-pass two-way automaton  $\mathcal{A}$ . By Theorem 2 it suffices to show that  $L(\mathcal{A})$  satisfies the lattice identity  $z(xy)^\omega \leftrightarrow z(xy)^\omega x$ . The *leaving state* of  $u$  is the state of  $\mathcal{A}$  which on input  $u$  encounters the right end marker  $\triangleleft$  for the first time. Note that, since  $\mathcal{A}$  is complete and deterministic, there is a unique state with this property. Consider words  $x, y, z \in A^*$ . There are only finitely many strongly connected components of  $\mathcal{A}$ . Consequently the pigeonhole principle yields an integer  $n$  such that the leaving states of  $z(xy)^n$  and of  $z(xy)^{n+1}$  are in the same strongly connected component. Hence, the same is true for the leaving states  $p$  of  $z(xy)^n$  and  $q$  of  $z(xy)^n x$ . Since  $\mathcal{A}$  is weak,  $p$  is final if and only if  $q$  is; and because  $\mathcal{A}$  is a one-pass automaton we have  $z(xy)^n \in L(\mathcal{A})$  if and only if  $z(xy)^n x \in L(\mathcal{A})$ . This establishes the lattice identity.  $\square$

Not every deterministic one-pass two-way automaton recognizing a Boolean combination of right ideals needs to be weak. Therefore, the equivalence of (1) and (4) in Theorem 12 does not follow from Proposition 14. Also note that the analogue of Proposition 14 does not work for right ideals (resp. prefix-closed languages) and deterministic flip (resp. fully accepting) one-pass two-way automata since deterministic two-way automata can also reject an input by an infinite cycle in its computation.

As for one-way automata in Section 4, we get right ideals in  $\mathcal{DA}$  if the recognizing automaton is a flip automaton. For a *flip automaton*, a transition  $z \xrightarrow{a} z'$  with final state  $z$  implies that  $z'$  is final. As an intermediate step, we get a characterization in terms of unambiguous monomials.

**Theorem 15** *Let  $L \subseteq A^*$ . The following are equivalent:*

1.  $L \in \mathcal{DA}(A^*)$  is a right ideal.
2.  $L$  is a finite union of unambiguous monomials  $A_1^* a_1 \cdots A_k^* a_k A^*$ .
3.  $L$  is recognized by a complete flip one-pass po2dfa.

*Proof.* We first show (1)  $\Rightarrow$  (3). Suppose  $L \in \mathcal{DA}(A^*)$  is a right ideal of  $A^*$ . By Theorem 12 there exists a complete one-pass po2dfa  $\mathcal{A}$  which recognizes  $L$ . We show how to obtain an equivalent automaton  $\mathcal{B}$  which is a flip automaton. Let us say that, during a computation, a deterministic automaton is in *progress mode* if after the next transition is taken, the automaton scans a position which has not been scanned before. The idea is that we need to change into a final state only when in progress mode. Note that for acceptance, the crucial transition of a (one-pass) two-way automaton is always made in progress mode. Moreover, consider an input  $uav$  and suppose  $\mathcal{A}$  scans position  $|ua|$  in progress mode and performs a transition into a final state. Then the prefix  $ua$  is accepted because  $\mathcal{A}$  is a one-pass automaton (note that this would not hold if  $\mathcal{A}$  has already seen some prefix of  $v$  during the computation). Since  $L$  is a right ideal, all words in  $uaA^*$  are accepted. This shows that if in progress mode a transition into a final state is made, then we can directly go into a final, right-moving sink state without changing the language. In total this yields a complete one-pass po2dfa which is a flip automaton. It remains to show that we can simulate  $\mathcal{A}$  in such a way that the simulation is aware of when it is in progress mode.

Assume that  $\mathcal{A}$  is leaving progress mode. This can only happen by a transition to a left-moving state. Suppose the input is factorized as  $u = u_1 a_1 \cdots u_m a_m u'$  where the  $a_i$ 's correspond to the positions where a state change happened while in progress mode. Note that  $a_m$  corresponds to the position scanned before taking the transition because a state

change is necessary to leave progress mode. Now since  $\mathcal{A}$  is deterministic, we see  $a_i \notin \text{alph}(u_i)$  for all  $i$ . Moreover since  $\mathcal{A}$  is partially ordered,  $m$  is bounded by the number of states of  $\mathcal{A}$ . Therefore, the simulation can store the word  $v = a_1 \cdots a_m$  in a stack of letters with bounded depth in its state space. Using the automaton  $\mathcal{C}$  of Lemma 10 with  $v = a_1 \cdots a_m$ , we can simulate  $\mathcal{A}$  in the subsequent non-progressing phase and recognize when we are scanning the frontier  $a_m$  of progress again. The automaton is complete and thus there eventually is a transition trespassing the position corresponding to  $a_m$ . This is when the simulation switches back to progress mode. Back in progress mode, the simulation organizes the stack by pushing the currently scanned letter to the stack if it causes a state change.

(3)  $\Rightarrow$  (2): Let  $L = L(\mathcal{A})$  for a complete one-pass po2dfa which is a flip automaton. For every  $u \in L(\mathcal{A})$  we construct an unambiguous monomial  $P(u) = A_1^* a_1 \cdots A_k^* a_k A^*$  such that  $u \in P(u) \subseteq L(\mathcal{A})$  and  $k$  bounded by the number of states of  $\mathcal{A}$ . Since there are only finitely many such monomials, we have  $L(\mathcal{A}) = \bigcup_{u \in L} P(u)$  and this union is finite.

To construct  $P(u)$  consider  $u \in L(\mathcal{A})$  and fix an accepting computation of  $\mathcal{A}$  on  $u$ . Consider the factorization  $u = u_1 a_1 \cdots u_k a_k u'$  where the  $a_i$ 's correspond to state changes. Let  $P(u) = A_1^* a_1 \cdots A_k^* a_k A^*$  with  $A_i = \text{alph}(u_i)$ . Trivially,  $u \in P(u)$  and  $k$  is bounded by the number of states of  $\mathcal{A}$ . Moreover,  $P(u)$  is unambiguous because  $\mathcal{A}$  is deterministic. It remains to show  $P(u) \subseteq L(\mathcal{A})$ . Suppose  $\mathcal{A}$  is in some state  $z$  while scanning a  $b$ -position of  $u_i$ . By construction there is no state change with the next transition, *i.e.*, there is a loop  $z \xrightarrow{b} z$  in  $\mathcal{A}$ . Consider some word  $v \in P(u)$  and factorize  $v = v_1 a_1 \cdots v_k a_k v'$  with  $v_i \in A_i^*$ . By construction, there exists a run of  $\mathcal{A}$  on  $v$  which eventually trespasses  $a_k$  into a final right-moving state. Then since  $\mathcal{A}$  is a complete flip automaton, no matter what comes beyond the  $a_k$  can remedy acceptance. This shows that every  $v \in P(u)$  is accepted.

(2)  $\Rightarrow$  (1): Every union of unambiguous monomials is in  $\mathcal{DA}$ , *cf.* [17, 4]. By Proposition 1, we see that  $L$  is a right ideal.  $\square$

Note that property (2) in Theorem 15 states that unambiguity of monomials and the ideal property can be achieved simultaneously, which is non-trivial. A two-way automaton is *fully accepting* if all its states are final. As for one-way automata, this yields prefix-closed languages (at least for  $\mathcal{DA}$ ). The following result for prefix-closed languages is an immediate corollary of Theorem 15.

**Corollary 16** *Let  $L \subseteq A^*$ . The following are equivalent:*

1.  $L \in \mathcal{DA}(A^*)$  is prefix-closed.
2.  $L$  is recognized by a fully accepting one-pass po2dfa.  $\square$

*Proof.* (1)  $\Rightarrow$  (2): Let  $L \in \mathcal{DA}(A^*)$  be prefix-closed. The complement  $A^* \setminus L$  is a right ideal in  $\mathcal{DA}(A^*)$  and thus Theorem 15 yields a complete one-pass po2dfa  $\mathcal{A} = (Z, A, \delta, x_0, F)$  which is flip and recognizes  $A^* \setminus L$ . We can assume  $x_0 \notin F$  since otherwise  $\varepsilon \in A^* \setminus L$  and thus  $A^* \setminus L = A^*$  and  $L = \emptyset$  (and for  $L = \emptyset$  we allow the empty automaton). Let  $\mathcal{A}' = (Z \setminus F, A, \delta', x_0, Z \setminus F)$  be the deterministic one-pass po2-automaton obtained from  $\mathcal{A}$  by restricting the states to  $Z \setminus F$ , *i.e.*, the transition relation  $\delta'$  is given by  $z \xrightarrow{a} z'$  in  $\mathcal{A}'$  if  $z, z' \in Z \setminus F$  and  $z \xrightarrow{a} z'$  in  $\mathcal{A}$ . Clearly,  $\mathcal{A}'$  is fully accepting and a straightforward verification yields  $L(\mathcal{A}') = A^* \setminus L(\mathcal{A})$ .

(2)  $\Rightarrow$  (1): Suppose  $L = L(\mathcal{A})$  for a fully accepting one-pass po2dfa  $\mathcal{A} = (Z, A, \delta, x_0, Z)$ . Let  $\mathcal{A}' = (Z \cup \{x_f\}, A, \delta', x_0, \{x_f\})$  where  $x_f$  is a new right-moving sink state, *i.e.*,  $\delta'$  extends  $\delta$  with transitions  $z \xrightarrow{a} x_f$  for  $z \in Z \cup \{x_f\}$  if there exists no  $z' \in Z$  such that  $(z, a, z') \in \delta$ . Then  $\mathcal{A}'$  is a complete one-pass flip po2dfa and  $L(\mathcal{A}')$  is a right ideal in  $\mathcal{DA}(A^*)$  by Theorem 15. Since  $L(\mathcal{A}) = A^* \setminus L(\mathcal{A}')$ , we see that  $L \in \mathcal{DA}(A^*)$  is prefix-closed.  $\square$



**Acknowledgments.** We thank the anonymous referees for several suggestions which helped to improve the presentation of the paper, and we are also grateful for bringing to our attention the works of Avgustinovich and Frid [1] and of Paz and Peleg [11].

## References

- [1] S. V. Avgustinovich and A. E. Frid. Canonical decomposition of a regular factorial language. In: *CSR 2006*, LNCS, pp. 18–22. Springer, Heidelberg, 2006.
- [2] D. Beauquier and J.-É. Pin. Languages and scanners. *Theor. Comput. Sci.*, 84(1):3–21, 1991.
- [3] L. Dartois, M. Kufleitner, and A. Lauser. Rankers over infinite words. In: *DLT 2010*, vol. 6224 of *LNCS*, pp. 148–159. Springer, 2010.
- [4] V. Diekert, P. Gastin, and M. Kufleitner. A survey on small fragments of first-order logic over finite words. *Int. J. Found. Comput. Sci.*, 19(3):513–548, 2008.
- [5] V. Diekert and M. Kufleitner. Fragments of first-order logic over infinite words. *Theory Comput. Syst.*, 48:486–516, 2011.
- [6] M. Gehrke, S. Grigorieff, and J.-É. Pin. Duality and equational theory of regular languages. In: *ICALP 2008*, vol. 5126 of *LNCS*, pp. 246–257. Springer, 2008.
- [7] M. Kufleitner and A. Lauser. Around dot-depth one. In: *AFL 2011*, pp. 255–269, 2011.
- [8] M. Kufleitner and A. Lauser. Partially ordered two-way Büchi automata. *Int. J. Found. Comput. Sci.*, 22(8):1861–1876, 2011.
- [9] R. McNaughton and S. Papert. *Counter-Free Automata*. The MIT Press, 1971.
- [10] D. E. Muller, A. Saoudi, and P. E. Schupp. Alternating automata, the weak monadic theory of the tree, and its complexity. In: *ICALP 1986*, vol. 226 of *LNCS*, pp. 275–283. Springer, 1986.
- [11] A. Paz and B. Peleg. Ultimate-definite and symmetric-definite events and automata. *J. Assoc. Comput. Mach.*, 12(3):399–410, 1965.
- [12] D. Perrin and J.-É. Pin. *Infinite words*, vol. 141 of *Pure and Applied Mathematics*. Elsevier, 2004.
- [13] J.-É. Pin and P. Weil. Polynomial closure and unambiguous product. *Theory Comput. Syst.*, 30(4):383–422, 1997.
- [14] Th. Schwentick, D. Thérien, and H. Vollmer. Partially-ordered two-way automata: A new characterization of DA. In: *DLT 2001*, vol. 2295 of *LNCS*, pp. 239–250. Springer, 2001.
- [15] L. Staiger.  $\omega$ -languages. In: *Handbook of Formal Languages*, vol. 3, pp. 339–387. Springer, 1997.
- [16] L. Staiger and K. W. Wagner. Automatentheoretische und automatenfreie Charakterisierungen topologischer Klassen regulärer Folgenmengen. *Elektron. Inform.-verarb. Kybernetik*, 10(7):379–392, 1974.
- [17] P. Tesson and D. Thérien. Diamonds are forever: The variety DA. In: *Semigroups, Algorithms, Automata and Languages 2001*, pp. 475–500. World Scientific, 2002.
- [18] W. Thomas. Automata on infinite objects. In: *Handbook of Theoretical Computer Science*, ch. 4, pp. 133–191. Elsevier, 1990.